MIXED ELEMENTS IN THE BOUNDARY THEORY

A. Martin, I. Rodriguez and E. Alarcón

E.T.S.I.I. de Madrid, Spain

1. INTRODUCTION

The development of (static and dynamics) programs with constant and linear elements has shown good behaviour.

It seems so natural to combine both advantages so that the results will not be affected by local distortions.

This paper will be dedicated to presenting the research of mixed elements and the way to solve the over-determination that appears in some cases.

Although all the study has been done with the potential theory, its application to elastic problems is straightforward.

2. BASES

The general formulation for mixed elements on the boundary theory is similar to the normal development given to the boundary problems.

The study is based on the integral equation defined on the boundary

$$C(1)\ \phi(1) + \sum_{k=1}^{N} \int_{\partial D} \phi_k \left(\frac{\partial \psi}{\partial n}\right)(1)\ ds_k =$$

$$= \sum_{k=1}^{N} \int_{\partial D} \psi_k\ q_k\ ds_k \qquad (1)$$

In this particular case, it will be considered a linear variation for the potential function along the element

$$\phi_k(\xi) = \left[N_1(\xi),\ N_2(\xi)\right] \begin{Bmatrix} \phi_k(k) \\ \phi_k(k+1) \end{Bmatrix}$$

and a constant variation for its flux q_k along the element.

The shape functions are:

$$N_1(\xi) = \tfrac{1}{2}(1-\xi) \qquad\qquad -1 \le \xi \le 1$$

$$N_2(\xi) = \tfrac{1}{2}(1+\xi)$$

The function ψ for 2D is:

$$\psi = \frac{1}{2\pi} \operatorname{Ln} \frac{1}{r}$$

Introducing the boundary conditions in the integral equation (1)

$$c(\ell)\ \phi(\ell) + \sum_{k=1}^{N} \int_{\partial D_k} \left[N_1,\ N_2\right] \left(\frac{\partial\psi}{\partial n}\right)_k \begin{Bmatrix} \phi_k(k) \\ \phi_k(k+1) \end{Bmatrix} ds_k =$$

$$= \sum_{k=1}^{N} q_k \int_{\partial D_k} \psi_k\ ds_k$$

This can be written:

$$c(\ell)\ \phi(\ell) + \sum_{k=1}^{N} \left[A_1,\ A_2\right] \begin{Bmatrix} \phi_k(k) \\ \phi_k(k+1) \end{Bmatrix} = \sum_{k=1}^{N} q_k\ G_{ik}$$

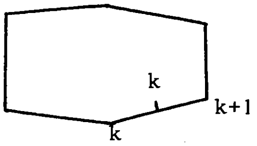
in which integration constants A_1, A_2, G_{ik} are:

$$A_1 = \int_{\partial D_k} N_1(\xi) \left(\frac{\partial\psi}{\partial n}\right)_k ds_k \ ;\ A_2 = \int_{\partial D_k} N_2(\xi) \left(\frac{\partial\psi}{\partial n}\right)_k ds_k \ ;\ G_{ik} = \int_{\partial D_k} \psi_k ds_k$$

As it could be seen, the equation (2) can be applied on any point ℓ where we have an unknown quantity, originating as many equations as unknown quantities; this point can be the node or the center of an element depending on the data.

3. INTEGRAL EQUATION TREATMENT

On a given problem, first we will discretize the boundary and define the boundary conditions.



To establish the linear variation of the potential function along the element, its value must be defined on two points, the nodes: $\phi_k^{(k)}$ and $\phi_k^{(k+1)}$ (the subindex represents element, the superindex the node) to establish the constant variation of the flux, its value must be defined on one single point, the center of the element; q_k.

Further on, the generation of the equations system will be discussed depending on the different cases.

4. DETERMINATION OF INTEGRATION CONSTANTS

If the integration is going to be done from an element, k, on which the potential function ϕ is unknown the integration will be done from the k node.

Integrating from the node to an element not continuous to it, the constants will be calculated with numerical methods.

$$A_1 = \frac{L_k}{4} \sum_{k=1}^{4} (1-\xi_i) \frac{D}{r_i^2} \omega_i$$

$$A_2 = -\frac{L_k}{4} \sum_{k=1}^{4} (1+\xi_i) \frac{D}{r_i^2} \omega_i$$

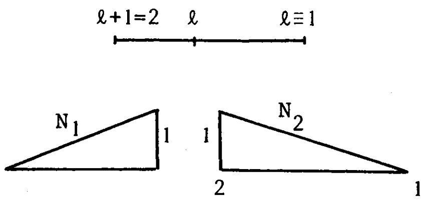
$$G_{ik} = \frac{L_k}{2} \sum_{k=1} \mathrm{Ln} \frac{1}{r_i} \omega_i$$

It is not possible to use this expression for contiguous elements because r can be zero.

Using analytical methods, an interesting relation between the constant can be found.

Analysing the values of A_1 and A_2 calculated from the ℓ node to the ℓ element.

$$A_1 = \int_{\partial D_L} N_1(\xi) \left(\frac{\partial\psi}{\partial n}\right) \frac{L_\ell}{2} \, d\xi$$



it is known that

$$\partial\psi/\partial n = D/r^2$$

It is easy to find that $\partial\psi/\partial n = 0$ along the element except on the point 1 where it is undetermined.

$N_1 = 1$ on 1 so A_1 is undetermined

$N_2 = 0$ on 1 so $A_2 = 0$

If we analyze the values of A_1 and A_2 integrating from node ℓ to element $\ell-1$ we get

$A = 0$

A_2 undetermined.

Applying the integral equation from the ℓ node, A_1 calculated on the $\ell-1$ element and A_2 on the ℓ element are zero, and A_1 on the ℓ element and A_2 on the $\ell-1$ element are undetermined; the constants are multiplying the value of the function ϕ on the ℓ node.

If we apply a constant potential ϕ, (2) becomes

$$c(\ell)\ \phi(\ell) + \sum_{k=1}^{N} (A_1, A_2) \begin{Bmatrix} \phi_k(k) \\ \phi_k(k+1) \end{Bmatrix} = 0$$

This expression leads to

$$H_{\ell k}\ \phi_k = 0$$

where

$$H_{\ell k} = A_2 \text{ (on k-1 element)} + A_1 \text{ (on k element)}$$

$$H_{\ell\ell} = A_2 \text{ (on } \ell\text{-1 element)} + A_1 \text{ (on } \ell \text{ element)} + c(\ell)$$

This also allows the calculation of the unknown constant $c(\ell)$.

G calculation:

$$G_{\ell\ell} = \int_{\partial D_\ell} \psi_\ell\ ds_\ell = \int \mathrm{Ln}\,\frac{1}{r}\,dr$$

$$G_{\ell\ell} = L_\ell\left(1 + \mathrm{Ln}\,\frac{1}{L_\ell}\right)$$

If the integration is going to be done from a k element on which the unknown is the flux q_k, the integration will be done from the center of the element.

Integrating from the k element to another element different from k the integral will be numerically evaluated.

$$A_1 = \frac{L_k}{4} \sum_{\ell=1}^{4} (1-\xi_i) \frac{}{r_i^2} \omega_i$$

$$A_2 = -\frac{L_k}{4} \sum_{\ell=1}^{4} (1+\xi_i) \frac{D}{r_i^2} \omega_i$$

$$G_{\ell k} = \frac{L_k}{2} \sum_{\ell=1}^{4} (\mathrm{Ln}\,\frac{1}{r_i}) \omega_i$$

This is different from the case before exposed on the definition of r_i.

A and A_2 are both undetermined but it can be demonstrated that A and A_2 are equal, if we integrate from the center to its own element.

$$\left.\begin{aligned} A_1 &= \int_{\partial D_k} N_1(\xi) \frac{\partial\psi}{\partial n}\, ds_k \\ A_2 &= \int_{\partial D_k} N_2(\xi) \frac{\partial\psi}{\partial n}\, ds_k \end{aligned}\right\} A_1 - A_2 = \int_{\partial D_k} (N_1 - N_2) \frac{\partial\psi}{\partial n}\, ds_k$$

$\partial\psi/\partial n$ is equal zero on the element except on the center of the element on which its value is undetermined.

ξ varies between -1 and 1, and it is zero in the center of the element.

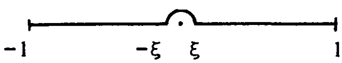
To save the undetermination we avoid that point integrating along a circumference with radius ξ, along the domain ∂D_ξ

$$A_1 - A_2 = \int_{\partial D_k - \partial D_\xi} (N_1 - N_2) \frac{\partial\psi}{\partial n}\, ds_k + \int_{\partial D_\xi} (N_1 - N_2) \frac{\partial\psi}{\partial n}\, ds_k$$

The first integral is zero because $\frac{\partial\psi}{\partial n} = 0$ on $\partial D_k - \partial D_\xi$

In the second integral

$$\psi = \frac{1}{2\pi}\, \mathrm{Ln}\, \frac{1}{\xi}$$



$$\frac{\partial\psi}{\partial n} = \frac{\partial\psi}{\partial\xi} \cdot \frac{\partial\xi}{\partial n} = \frac{\partial\psi}{\partial\xi} = -\frac{1}{2\pi\xi}$$

$$N_1 - N_2 = \xi$$

$$ds_\xi = 2\pi d\xi$$

$$A_1 - A_2 = \lim_{\xi \to o} \int_{-\xi}^{\xi} (N_1 - N_2) \frac{\partial\psi}{\partial n} ds_\xi = 0$$

$$A_1 = A_2$$

If we introduce a constant potential ϕ as has been done before for the integration from a node to contiguous elements we obtain:

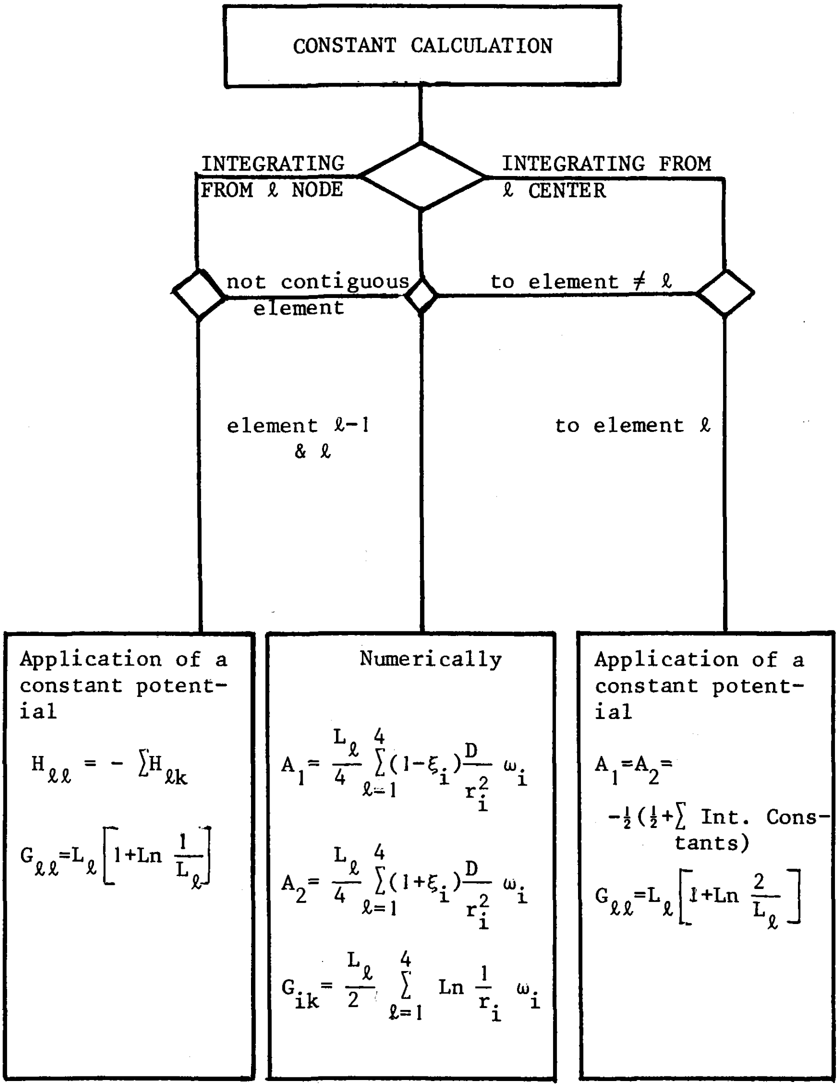
$A = A_2 = -\frac{1}{2}(C(\ell) + \sum$ integration constants)

$C = \frac{1}{2}$ because we have a regular boundary

$G_{\ell\ell}$ is calculated directly

$$G_{\ell\ell} = L_\ell \left[1 + \text{Ln}\,\frac{2}{L_\ell} \right]$$

In summary the different cases are:



Operating with the equation (2) we reach a matrix equation such as

$$\underline{P} \cdot \underline{X} = \underline{F}$$

in which generation will be shown.

5. BOUNDARY CONDITIONS

Circulating along the element from node to node in the positive way the flux will be considered known if it is known in the next element.

On the k node the different possibilities are:

1. q_k known, ϕ_k unknown, the integration will be done from the node.
2. ϕ_k known, q_k unknown, the integration will be done from the center of the k element.
3. ϕ_k and q_k known, possibility that this will not generate any equation.

Integrating from k node or center of k element to the element.

- On ℓ node q_ℓ and ϕ_ℓ are known

$$P_{k\ell} = P_{k\ell} + A_1$$

$$F_k = F_k + G_{\ell k}\, q_k$$

$$\phi_\ell = X_\ell$$

- ϕ_ℓ known and q_k unknown

$$P_{\ell k} = -G_{\ell k}$$

$$F_\ell = F_\ell - A_1 \phi_\ell$$

$$X_\ell = -q_\ell$$

- ϕ_ℓ and q_ℓ known

$$F_k = F_k - A_1 \phi_\ell + G_{\ell k}\, q_\ell$$

$\ell+1$ node

- q_ℓ is known and $\phi_{\ell+1}$ is unknown

$$P_{k,\ell+1} = P_{k,\ell+1} + A_2$$

q unknown and ϕ known or both known

$$R_k = F_k - A_2 \, \phi_{\ell+1}$$

6. CONCLUSIONS

In this paper the Boundary Methode for mixed elements has been developed, calculating the integration constants of the formulation.

A procedure has been developed that avoids the inconvenience of the general formulations which allows practical solution of all the possible cases on the boundary.

REFERENCES

1. F. Paris, A. Martin and E. Alarcon. "Boundary elements in potential and elasticity theory". Computers and Structures, Vol. 10. pp. 351-362, 1979.

2. F. Paris. El metodo de los elementos de contorno en la teoria del potencial y de la elasticidad. Ph.D. Thesis. E.T.S. Ingenieros Industriales, Madrid, 1979.

3. C.A. Brebbia. "The Boundary Element Method for Engineers" Pentech Press, 1978.

4. F.J. Rizzo. "An integral equation approach to boundary value problems in classical elastostatics". Quarterly of Applied Mathematics, 1967.